Complex Divisors on Algebraic Curves and Some Applications to String Theory[*]

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This talk presents some new notions of the theory of complex algebraic curves which have appeared as algebraic tools in string theory (see [4] for more details). In a sense, we have materialized non-existed complex powers of invertible sheaves on algebraic curves introduced at the level of the Atiyah algebras of invertible sheaves by Beilinson and Schechtman [1]. The *Atiyah algebra* $A_{\mathcal{L}}$ in the case of an invertible sheaf $\mathcal{L}$ over a complete complex algebraic curve X is just the sheaf of differential operators of order ≤ 1 on $\mathcal{L}$. There takes place the exact sequence $0 \to \mathcal{O} \to A_{\mathcal{L}} \to T \to 0$, where $\mathcal{O}$ is the structural sheaf and T is the tangent sheaf of X. The diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{O} & \to & A_{\mathcal{L}} & \to & T & \to 0 \\ & c\downarrow & & \downarrow & & \| & \\ 0 \to & \mathcal{O} & \to & ? & \to & T & \to 0 \end{array},$$

where c is the operator of multiplication by $c \in \mathbb{C}$, as usual, can be completed to commutative with a sheaf $A_{\mathcal{L}^c}$ interpreted as the Atiyah algebra of a (nonexisting) invertible sheaf $\mathcal{L}^c$, $\mathcal{L}$ to the power c. The Atiyah algebra keeps

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incomplete information about its invertible sheaf, because an isomorphism $A_{\mathcal{L}} \xrightarrow{\sim} A_{\mathcal{O}}$ implies the existence of a canonical flat holomorphic connection on $\mathcal{L}$, but not the triviality of $\mathcal{L}$. It is remarkable that using Atiyah algebras one can apply local arguments to Riemann-Roch type theorems.

Here we deal with really existing invertible sheaves, corresponding to divisors with complex coefficients, but having integral degree. In particular, every invertible sheaf of degree 0 can be raised to a complex power.

1 Complex Divisors

Let X be a complete complex curve of genus g, with a fixed ordered set $\mathfrak{m} = \{Q_1, \ldots, Q_n\}$ of n distinct points on X and a closed disk B, considered up to an isotopy in $X \setminus \mathfrak{m}$, such that $\mathfrak{m} \subset B$. A *complex divisor* is a formal sum

$$D = \sum_{P \in X} n_P \cdot P,$$

where

$$n_P \in \begin{cases} \mathbb{C} & \text{for } P \in \mathfrak{m}, \\ \mathbb{Z} & \text{otherwise}, \end{cases}$$

$$\deg D := \sum_{P \in X} n_P \ \in \mathbb{Z},$$

and only a finite number of $n_P \neq 0$. The corresponding *group of complex divisors* is denoted by $\mathrm{Div}\,(X, \mathfrak{m}, B)$.

This definition may be not interesting itself, but it leads to a new class of invertible sheaves over X.

2 Multiple Valued Meromorphic Functions

Let $p : \widetilde{X \setminus \mathfrak{m}} \to X \setminus \mathfrak{m}$ be the universal covering with the complex structure lifted from the base. Denote by H the kernel of the natural epimorphism $\pi_1(X \setminus \mathfrak{m}) \to \pi_1(X)$ determined by the embedding $X \setminus \mathfrak{m} \hookrightarrow X$:

$$1 \to H \to \pi_1(X \setminus \mathfrak{m}) \to \pi_1(X) \to 1.$$

We will call a holomorphic function ϕ on $\widetilde{X \setminus \mathfrak{m}}$ (more precisely, a section of the sheaf $p_* \mathcal{O}_{\widetilde{X \setminus \mathfrak{m}}}$) a *multiple valued holomorpic function* on X, if

1. ϕ is $\pi_1(X \setminus B)$-invariant (i.e., ϕ is single valued outside B),

2. for every $\sigma \in H$ $\phi^\sigma = f_\sigma \cdot \phi$, where $\phi^\sigma(z) := \phi(\sigma z)$, and f_σ is a constant (f_σ is called the *multiplicator*),

3. the branches of ϕ, as the branches of a multiple valued analytic function on X, have only removable singularities in $\mathfrak{m}$, that is for any $Q_i \in \mathfrak{m}$ and any sequence $\{a_m\}$ in a domain of univalence in $\widetilde{X \setminus \mathfrak{m}}$, such that $p(a_m) \to Q_i$ as $m \to \infty$, there exists a limit $\lim_{m\to\infty} \phi(a_m)$, depending only on Q_i:
$$\phi(Q_i) := \lim_{m\to\infty} \phi(a_m).$$

Denote by $\mathcal{O}'$ the sheaf of holomorphic multiple valued functions on X. Denote the corresponding sheaf of fields of fractions by $\mathcal{M}'$. We will call sections of $\mathcal{M}'$ *multiple valued meromorphic functions* on X. The following simple lemma describes the local behavior of such functions.

Lemma 1 *Let z be a holomorphic coordinate on X near $Q_i \in \mathfrak{m} \subset X$.*

1. If $\phi \in \mathcal{O}'$, then either

$$\phi(z) = z^A \cdot \sum_{j=0}^{\infty} \alpha_j z^j, \ \textit{where } 0 < \Re A \le 1,$$

or

$$\phi(z) = \sum_{j=0}^{\infty} \alpha_j z^j.$$

2. If $\phi \in \mathcal{M}'$, then

$$\phi(z) = z^A \cdot \sum_{j=n_0}^{\infty} \alpha_j z^j, \ \textit{where } 0 \le \Re A < 1. \square$$

Note. One should remember that these expansions may also get monodromy at other points $Q_i \in \mathfrak{m}$.

Definition. The number $A + n_0$ is called the order $\mathrm{ord}_{Q_i}\phi$ of the multiple valued holomorphic function ϕ at the singular point Q_i.

Let $\phi \in \Gamma(X, \mathcal{M}')$ be a globally defined multiple valued holomorphic function on X. Then $\sum_{P\in X} \mathrm{ord}_P\phi = 0$, because $\mathrm{d}\log\phi$ is a differential of the

third kind on X and the sum of its residues vanishes.

Definition. A divisor of the type

$$\operatorname{div} \phi := \sum_{P \in X} \operatorname{ord}_P \phi \cdot P$$

is called principal.

Define the *group* $\mathrm{Cl}\,(X, \mathfrak{m}, B)$ *of classes of complex divisors* as the quotient-group of the group $\mathrm{Div}\,(X, \mathfrak{m}, B)$ by the subgroup of principal divisors.

3 Complex Divisors and Invertible Sheaves

Proposition 1 *1. The group* $\mathrm{Cl}\,(X, \mathfrak{m}, B)$ *is isomorphic to the group* $\mathrm{Cl}\,(X)$ *of classes of ordinary (integral) divisors on* X*.*

2. The group $\mathrm{Div}\,(X, \mathfrak{m}, B)$ *is isomorphic to the group of invertible* $\mathcal{O}$*-submodules in* $\mathcal{M}'$*.*

Proof. Part 1 evidently follows from 2, so let us prove 2. Choose a covering of X with two open subsets $U_1 := \{$a δ-neighborhood of B for small $\delta > 0$, $U_2 := X \setminus B$, and given complex divisor $D = \sum n_P \cdot P$ take a multiple valued meromorphic function f_1 on U_1, such that $\operatorname{ord}_P f_1 = n_P$ for $P \in U_1$, and a multiple valued meromorphic function f_2 on U_2, such that $\operatorname{ord}_P f_2 = n_P$ for $P \in U_2$ and $f_2^{\gamma_i} = \exp(2\pi\sqrt{-1} \cdot n_{Q_i}) \cdot f_2,\ i = 1, \ldots, n$, where γ_i is a loop in B containing the single point Q_i. Then f_1/f_2 is a single valued nonzero holomorphic function on $U_1 \cap U_2$, i.e., $f_1/f_2 \in \Gamma(U_1 \cap U_2, \mathcal{O}^*)$, and it determines an $\mathcal{O}$-submodule $\mathcal{O}(D)$ in $\mathcal{M}'$ having f_2/f_1 as the glueing function. Thus, $\mathcal{O}(D)$ is an ordinary invertible sheaf on X. $\square$

4 The Weil-Deligne Pairing

Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two invertible $\mathcal{O}$-modules. [They may well be $\mathcal{O}$-submodules in $\mathcal{M}'$]. Define a complex vector space $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ as the space generated by the expressions

$$\langle l_1, l_2 \rangle, \tag{1}$$

where l_1 and l_2 are single valued (i.e., having integral divisors) meromorphic sections of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, with nonintersecting divisors. We place the following relations on the symbols (1):

$$\langle f \cdot l_1, l_2 \rangle = f(\operatorname{div} l_2) \cdot \langle l_1, l_2 \rangle,$$

$$\langle l_1, g \cdot l_2 \rangle = g(\operatorname{div} l_1) \cdot \langle l_1, l_2 \rangle,$$

where f and g are single valued meromorphic functions such that $f(\operatorname{div} l_2) := \prod_{P \in X} f(P)^{\operatorname{ord}_P l_2} \neq 0, \infty$ in the former formula and $g(\operatorname{div} l_1) := \prod_{P \in X} g(P)^{\operatorname{ord}_P l_1} \neq 0, \infty$ in the latter. The correctness of this definition is provided by Weil's reciprocity law:

$$f(\operatorname{div} g) = g(\operatorname{div} f).$$

One can easily see that the space $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ is a one-dimensional complex vector space. We will call it the *Weil-Deligne pairing* of $\mathcal{L}_1$ and $\mathcal{L}_2$.

5 The Arakelov-Deligne Metric

Now, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Hermitian holomorphic line bundles. Then one can define a natural Hermitian metric on the space $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ (cf. Deligne [2]). That means that for any two sinle valued sections l_1, l_2 of $\mathcal{L}_1$ and $\mathcal{L}_2$ with nonintersecting divisors, there is defined a real number

$$\| \langle l_1, l_2 \rangle \| \in \mathbb{R}.$$

Below we define an analogous metric in a more general case, when l_1 and l_2 are not necessarily single valued, but of degree 0. We will use this construction in string theory later.

The definition is

$$\| \langle l_1, l_2 \rangle \| := \sqrt{\prod_i G^{\overline{n}_i}_{\operatorname{div} l_2}(P_i) \cdot G^{n_i}_{\overline{\operatorname{div} l_2}}(P_i)}, \tag{2}$$

where $\overline{\operatorname{div} l_2}$ means the divisor with complex conjugated coefficients, $\operatorname{div} l_1 = \sum_i n_i P_i$ and $G_D(z) := \exp g_D(z)$, $g_D(z)$ being the Green function of the divisor D, which is defined up to a constant similar to the case of integral D (for example, put $g_D(z) := \Re \int_{z_0}^{z} \omega_D$, where ω_D is the differential of the third

kind associated with D, cf. Lang [3]). The result does not depend on the choice of Green function, because we assume $\deg l_1 = \sum n_i = 0$. Moreover, the obtained symbol $\| \langle\ ,\ \rangle \|$ is symmetric:

$$\| \langle l_1, l_2 \rangle \| = \| \langle l_2, l_1 \rangle \| .$$

This can be observed from the formula

$$\| \langle l_1, l_2 \rangle \| = \sqrt{\prod_i G_{\operatorname{div} l_2}^{\overline{n}_i}(P_i) \cdot \prod_j G_{\operatorname{div} l_1}^{\overline{n}'_j}(P'_j)}, \tag{3}$$

where $\operatorname{div} l_2 = \sum_j n'_j P'_j$. In fact, $G_{\operatorname{div} l_1}(z) = \prod_i G_{P_i}^{n_i}(z)$ and $G_P(Q) = G_Q(P)$, so (3) is equivalent to (2). These arguments also imply the formula

$$\| \langle l_1, l_2 \rangle \| = \prod_{i,j} G_{P_i}^{\Re(n_i \overline{n}'_j)}(P'_j). \tag{4}$$

If l_1, l_2 and k are sections of Hermitian line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ and $\mathcal{K}$, then

$$\| \langle l_1 \otimes l_2, k \rangle \| = \| \langle l_1, k \rangle \| \cdot \| \langle l_2, k \rangle \|$$

whenever both sides are defined. There are some special properties of complex divisors: if $\operatorname{supp} D_1$, $\operatorname{supp} D_2 \subset \mathfrak{m}$, then

$$\| \langle \mathbf{1}_{\alpha D_1}, \mathbf{1}_{D_2} \rangle \| = \| \langle \mathbf{1}_{D_1}, \mathbf{1}_{D_2} \rangle \|^{\alpha} \quad \text{for } \alpha \in \mathbb{R}$$

and

$$\| \langle \mathbf{1}_{\alpha D_1}, \mathbf{1}_{D_2} \rangle \| = \| \langle \mathbf{1}_{D_1}, \mathbf{1}_{\overline{\alpha} D_2} \rangle \| \quad \text{for } \alpha \in \mathbb{C}.$$

Thereby, the symbol $\| \langle\ ,\ \rangle \|$ is Hermitian. More precisely, it is the modulus of the exponent of a Hermitian form on the vector space of complex divisors of degree 0 with support in $\mathfrak{m}$. This Hermitian form is easy to write out (cf. (4)):

$$\sum_{i,j} n_i \, \overline{n}'_j \, g_{Q_i}(Q_j).$$

6 The Deligne-Riemann-Roch Theorem

Let us consider an algebraic family of objects $(X, \mathfrak{m}, B)$ over a base S, i.e., a smooth projective morphism $\pi : X \to S$ of smooth complex algebraic

varieties with fiber being a connected complex curve, $\mathfrak{m}$ being the disjoint union of n regular sections of π and B varying continuously along S. Let D and D' be two families of complex divisors on $X \to S$, more generally, two invertible $\mathcal{O}$-submodules $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{M}'$. Suppose they are metrized as well as the sheaf Ω of relative 1-differentials along the fibers of π. Then the sheaves $\det \mathbb{R}\pi_* \mathcal{L}$ (the determinant sheaf) and $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ (the Weil-Deligne sheaf, whose fiber over a single curve X_s, $s \in S$, in the family is defined in Section 4) are defined. $\det \mathbb{R}\pi_* \mathcal{L}$ can be endowed with a Hermitian metric according to Quillen (see [2]), and $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ is metrized in Section 5. The following theorem is important for our string applications.

Theorem 1 (Deligne [2]) *There is a canonical isometry*

$$\det \mathbb{R}\pi_*(\mathcal{L})^2 \otimes \det \mathbb{R}\pi_*(\mathcal{O})^{-2} = \langle \mathcal{L} \otimes \Omega^*, \mathcal{L} \rangle . \Box$$

7 String Applications

The g-loop contribution to the string partition function can be reduced to the integral

$$Z_g := \int_{\mathcal{M}_g} \mathrm{d}\pi_g$$

of the *Polyakov measure* $\mathrm{d}\pi_g$ over the moduli space $\mathcal{M}_g$ of complete complex algebraic curves of genus g. The *Belavin-Knizhnik theorem* represents $\mathrm{d}\pi_g$ as the modulus squared

$$\mathrm{d}\pi_g = \mu_g \wedge \overline{\mu}_g$$

of a *Mumford form* μ_g, which is a section of the sheaf $\lambda_2 \otimes \lambda_1^{-13}$, where $\lambda_i := \det \mathbb{R}\pi_*(\Omega^{\otimes i})$, π being the universal curve $\pi : X \to \mathcal{M}_g$.

The *tachyon scattering amplitude* is the integral

$$A(g; \mathbf{p}_1, \dots, \mathbf{p}_n) := \int_{\mathcal{M}_{g,n}} \mathrm{d}\pi_{g,n},$$

where $\mathcal{M}_{g,n}$ is the moduli space of algebraic curves of genus g with n punctures and the measure $\mathrm{d}\pi_{g,n}$ is expressed in terms of determinants of Laplace operators and their Green functions. The vectors $\mathbf{p}_i$ on which the amplitude depends are regarded as momentum vectors at the scattering points, so they lie in the space-time of the critical dimension, which we identify with

$\mathbb{C}^{13}$ endowed with the standard Hermitian metric. These vectors satisfy the conditions:

1. $\sum_{i=1}^{n} \mathbf{p}_i = 0$ (the momentum conservation law).

2. The Hermitian square $(\mathbf{p}_i, \mathbf{p}_i)$ is equal to 1 for every i (the mass of tachyon is $\sqrt{-1}$).

Our application to string theory consists in proving the following analogue of the Belavin-Knizhnik theorem for string amplitudes.

Theorem 2

$$\mathrm{d}\pi_{g,n} = \mu_{g,n,B} \wedge \overline{\mu}_{g,n,B} / \parallel \mu_{g,n,B} \parallel^2,$$

where $\mu_{g,n,B}$ *is a local holomorphic section of the Hermitian line bundle* $\lambda_2 \otimes \lambda_1^{-13} \otimes (\bigotimes_{\nu=1}^{13} \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle)^{-1}$ *over the moduli space* $\mathcal{M}_{g,n,B}$ *of the data* $(X, Q_1, \ldots, Q_n, B)$*. Here* $D^\nu := \sum_{i=1}^{n} p_i^\nu \cdot Q_i$ *is the complex divisor with the momentum components as coefficients. The section* $\mu_{g,n,B}$ *is defined locally up to a holomorphic factor.* $\square$

References

[1] A. A. Beilinson and V. V. Schechtman. Determinant bundles and Virasoro algebras. *Commun. Math. Phys.*, 118:651–701, 1988.

[2] P. Deligne. Le determinant de la cohomologie. *Contemp. Math.*, 67:93–178, 1987.

[3] S. Lang. *Fundamentals of Diophantine Geometry.* Springer-Verlag, New York–Berlin, 1983.

[4] A. A. Voronov. A unified approach to string scattering amplitudes. *Commun. Math. Phys.*, 131:179–218, 1990. Erratum: 140:415–416, 1991.